State and Rate Dependent Friction Laws for Modeling High-Speed Frictional Slip at Metal-on-Metal Interfaces

Hamid Ullah
Department of Mechanical Engineering, NWFP University of Engineering & Technology, Peshawar 25000, Pakistan
Design and Manufacturing Engineering, School of Engineering and Technology, Asian Institute of Technology, Pathumthani 12120, Thailand
e-mail: hamid.ullah@ait.ac.th

M. A. Irfan
Department of Mechanical Engineering, NWFP University of Engineering & Technology, Peshawar 25000, Pakistan

V. Prakash
Department of Mechanical & Aerospace Engineering, Case Western Reserve University, Cleveland, OH 44106

1 Introduction

The nature of dynamic friction forces between two bodies in contact is a complex process and is affected by a long list of factors including the interface constitution, time scales of contact, the accumulated frictional slip, slip velocity, normal stress, inertia and thermal effects, roughness of the contacting surfaces, history of loading, and so on. Most tribologists will agree that friction at low slip speeds is due in large measure to local adhesion and shearing of regions in contact. In many cases the plastic strains obtained during sliding are very high and values exceeding plastic shear strains of 10−100% have been reported [1–3]. Besides the plastic deformation, in a sliding specimen, other important phenomena involve interactions with the counter-face material. These lead to changes not only in the microstructure but chemical composition as well, and the impact on both friction and wear can be dramatic [4]. Surface temperature between rubbing surfaces rises appreciably if the sliding speed is increased. This is because the major part of the work expended in overcoming the frictional resistance appears as heat in the regions of contact [5,6]. However, surface heating at the slip speeds commonly used in engineering practice is a transient and localized phenomenon, and does not have a great influence on the resultant friction mechanism. In marked contrast to this, at relatively high sliding speeds, the metal surfaces are subjected to intense frictional heating [7–11], which profoundly changes the state of the surface layers in which the sliding takes place [12].

In the present paper the applicability of state and rate dependent friction laws in describing the phenomena of high speed slip at metal-on-metal interfaces is investigated. For the purpose of model validation, results of plate-impact pressure-shear friction experiments were conducted by Irfan in 1998 and Irfan and Prakash in 2000 using a Ti6Al4V and Carpenter Hambden tool-steel tribo pair are employed. In these experiments high normal pressures (1–3 GPa) and slip speeds of approximately 50 m/s were attained during the high-speed slip event. Moreover, these experiments were designed to investigate the evolution of friction stress in response to step changes in normal pressure and also in the applied shear stress during the high-speed slip event. A step drop in normal pressure is observed to result in an exponential decay of the friction stress to a new steady-state characteristic of the current normal pressure and the current slip velocity. A step drop in applied shear stress is observed to lead to an initial drop in friction stress, which later increases toward a new steady-state friction stress level. In response to the step drop in applied shear stress the slip velocity initially increases and then decreases to a new steady-state level consistent with the new friction stress level. A modified rate and state dependent friction model that employs both velocity and normal stress dependent state variables is used to simulate the experimental results. A good correlation is found between the experimental results and the predictions of the proposed state and rate dependent friction model. [DOI: 10.1115/1.2401217]

Keywords: mathematical modeling, high speed frictional slip, plate impact friction experiments, state and rate dependent friction

Dynamic friction characteristics at high slip-speeds and/or high normal pressures have been investigated by several investigators. In some of the early works, Shugart’s [13] developed an experimental apparatus in which a rod was pressed against the rim of a rapidly revolving disk. However, any quantitative measurement of the coefficient of friction was hampered by effects of mechanical vibrations and severe frictional heating. Krafft [14] investigated high-speed friction in a series of ballistic tests where friction occurred between a rotating bullet and the steel target it penetrated. Post-examination of the tips of the projectile revealed the metal had melted in the penetration process. Using a high speed pin-on-disk test device, experiments were conducted [15–18]. The results of the experiment indicate that at large sliding velocities a reduction in the coefficient of friction is obtained when the normal pressure and/or the sliding velocity are increased. Montgomery and Bowden and Persson [8] spun a steel ball to a very high rotational speed and then grabbed it with other frictional samples or dropped it on another sample to achieve very high relative velocities (up to 800 m/s). However, the normal pressures were quite low (less than 0.015 MPa). In these experiments velocity weakening of the frictional force as a function of increasing relative velocities was observed. The rapid liberation of frictional energy resulted in a rise of surface temperature, in an extremely short time, to a value at or near the lower of the two melting points of the metals involved [20–22]. The high temperature was confined to a thin surface layer, and decreased very quickly as the distance from the surface increased. Higher pressures
(50–100 MPa) but much smaller slip velocities (~5 m/s) were obtained in experiments by Ogawa [23], who modified a split Hopkinson pressure bar to study impact friction by axially impinging the input tube on a rotating output tube. Rajagopalan et al. [24] modified the conventional torsional Kolsky bar apparatus to investigate high-speed friction at normal pressures of 50–100 MPa and slip velocities up to 10 m/s. Higher normal pressures (1–3 GPa) and higher slip velocities (2–60 m/s) were attained in the plate-impact pressure–shear friction experiments [25–28]. In the latter experiments interfacial temperatures of 800°C were obtained for a CH tool-steel Ti6Al4V tribo pair.

In order to model the friction behavior of sliding interfaces, a general class of frictional laws have been put forward that involves the introduction of state variables [29–33]. The basic assumptions underlying these state variable friction laws is that at the surface, at any instant of time, can be represented by a state. The state is assumed to be characterized by a collection of state variables, i.e., \( \theta = \theta_1, \theta_2, \theta_3, \ldots, \theta_n \), which may represent the surface roughness of the contacting plates, the average time of asperity contact, recent slip speeds, and surface dilatation/separation [34–36] or some other microstructural details. The usefulness of the state variable concept does not depend upon the physical interpretation of the state variables (like temperature or entropy in thermodynamics) though the discovery of such interpretation would add tremendously to the credence and usefulness of the theory. Within the realm of this theory, the frictional resistance \( \tau \) can be described if the instantaneous slip velocity \( V \), the normal pressure \( P \), and the state \( \theta \), such that \( \tau = f(P, V, \theta_1, \theta_2, \ldots, \theta_n) \). Then, this equation along with the evolutionary relations for the state variables \( \theta_i \)

\[
\frac{d\theta_i}{dt} = G_i(P, V, \theta_1, \theta_2, \ldots, \theta_n) \tag{1}
\]

provides a complete set of equations for describing the frictional process.

The major focus of the state-variable laws has been to describe the variation of frictional resistance to sliding under conditions of variable slip rate at constant normal pressure. Under these conditions, the resistance to sliding has been observed to be a function of the logarithm of the sliding velocity and exhibits a fading memory dependence on the slip history [29,30]. While these laws have been developed for rocks, their properties have recently been observed for a range of materials including lubricated steel on steel [37], teflon on steel, glass, plastic, and wood.

To date, the major focus in the laboratory has been to describe the variation of resistance to sliding under conditions of variable slip velocities at constant normal pressure. However in most dynamic slip situations the assumption of constant normal pressure and constant slip velocities at constant normal pressure. However in most dynamic slip situations the assumption of constant normal pressure and the slip velocity condition at the slip interface were varied by varying the impact velocity. By obtaining the history of the normal and shear tractions and the slip velocity from the experimental results, the behavior of the interface to sudden changes in normal pressure and the slip velocity was documented. A modified rate and state dependent friction model that employs both velocity and normal stress dependent state variables is used to simulate the experimental results. In the following sections the details of this state and rate dependent friction model and a comparison model prediction with the experimental results are presented.

2 Brief Description of the Plate-Impact Pressure–Shear Friction Experimental Configuration

The schematic of the plate-impact pressure–shear friction experiment is shown in Fig. 1. This configuration, which is a modification of the configurations used for studying the high strain rate shearing resistance of metals [41], involves the pressure–shear loading of a target with a flyer plate. The target and the flyer plates are chosen to represent the tribo-pair materials under investigation. The impacting plates are flat and parallel, and inclined relative to the direction of approach. Both the flyer and the target plates are unstrained initially. The target is at rest while the flyer is carried by a moving projectile at a known velocity \( V_0 \). Thus, the initial normal and transverse particle velocities \( u_0 \) and \( v_0 \) of the flyer are given by \( u_0 = V_0 \cos \theta \) and \( v_0 = V_0 \sin \theta \), where \( \theta \) is the skew angle.

At impact, both normal and transverse components of the flyer plate velocity are imposed on the target plate. Two stress waves propagate into the flyer and the target; a longitudinal wave propagating at the longitudinal wave speed \( c_1 \) and a transverse wave propagating at the shear wave speed \( c_2 \). A laser interferometry technique is used to measure the normal and transverse particle velocity components at a monitoring point on the rear surface of the target plate [42]. To investigate the dynamic sliding characteristics of frictional interfaces, the thickness of the flyer and the target plates are designed such that the time for longitudinal wave propagation through the thickness of the flyer plate is greater than the corresponding round-trip time of the shear waves in the target plate and the unloading waves generated at the lateral boundary. Under these conditions, when the longitudinal wave reflected from the free surface of the target plate arrives at the target/flyer interface, the normal pressure at the frictional interface is changed.

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**Fig. 1 Schematic of the pressure–shear friction experiment.** The impacting surfaces of the flyer and target plates represent the tribo-pair.
instantaneously. Since the longitudinal impedance of the flyer plate is less than the longitudinal impedance of the target plate, it results in a step drop in the applied normal pressure. Also, when the shear wave reflected from the target free surface arrives at the frictional interface it produces a new frictional state while maintaining the same normal pressure. Thus, three distinct dynamic frictional states are obtained in each experiment.

For the Ti6Al4V/CH steel tribo pair employed in the present experiments the duration of State 1 is approximately 1900 ns, followed by State 2 of duration 1600 ns and State 3 of duration 300 ns. During the transition from State 1 to State 2 the normal pressure is $P_1 > 0.33 P_2$. The corresponding normal stress, the friction (shear) stress, and the slip velocity in the three states can be related to the impact velocity, the skew angle of impact, the measured normal and transverse free surface particle velocities, and the longitudinal and shear impedances of the flyer and the target plates. Further details of the experimental facility, the procedure for conducting the experiments, and calculations for the normal and shear tractions and the slip velocity at the frictional interface can be found in Ref. [28].

### 3 Plate Impact Friction Experiments and the State and Rate Dependent Friction Model

Using the pressure–shear friction configuration a series of plate-impact experiments was conducted to investigate the high-speed frictional behavior for the Ti6Al4V and Carpenter Hampden tool steel tribo pair. Table 1 summarizes the two plate impact experiments employed for model validation in the present study. In these experiments Ti6Al4V was used as the flyer while the Carpenter Hampden tool-steel plate was used as the target. Additionally, the skew angle $\theta$ was kept at 35 deg. The frictional state at the tribo pair interface was controlled by varying the normal pressure (by varying the impact velocity), and/or the surface roughness of the impacting plates. Moreover, by appropriate selection of the flyer and target plate thicknesses the tribo-pair interface was subjected to step changes in normal pressure and step changes in applied shear stress.

Experiment Shot 9702 was designed to study the effect of normal pressure on interfacial frictional resistance of sliding interfaces. In view of this, the tribo pair surfaces were kept relatively smooth. For Shot 9702 the surface roughness of Ti6Al4V plate was $R_q = 0.09$, whereas the surface roughness of the Carpenter Hampden (CH) tool-steel plate was 0.02. The impact velocity for Shot 9702 was 95 m/s ($V = 1.292$ GPa). Experiment Shot 9704 was designed to study the effect of normal pressure on interfacial frictional behavior when the surface roughness of the CH tool-steel plate (the harder plate in the tribo pair) was increased substantially. In view of this, experiment Shot 9704 was conducted with a surface roughness of 0.11, while the surface roughness of the Ti6Al4V flyer plate was 0.09. The impact velocity for Shot 9704 was 80.5 m/s. The results of these experimental results are shown in Figs. 2–7, and will be described in more detail in Sec. 4.

The details of the state and rate dependent friction model used to simulate these experiments are provided next. Following Prakash [43], the shear stress at the frictional interface can be written as

$$\tau = f(V, \theta)g(P, \theta)$$

In Eq. (2) the functions $f$ and $g$ can be expressed in terms of the slip velocity, the normal pressure on the interface, and the state variables as

### Table 1 Summary of experimental results at Ti6 Al4V/CH steel interface

<table>
<thead>
<tr>
<th>Shot No.</th>
<th>Skew angle $\theta$ (deg)</th>
<th>Impact velocity $V$ (m/s)</th>
<th>Normal stress $R_q$ (GPa)</th>
<th>Ti6Al4V roughness $R_q$ ($\mu$m)</th>
<th>CH steel roughness $R_q$ ($\mu$m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>9702</td>
<td>35</td>
<td>95</td>
<td>1.292</td>
<td>0.07</td>
<td>0.09</td>
</tr>
<tr>
<td>9704</td>
<td>35</td>
<td>80.5</td>
<td>1.078</td>
<td>0.09</td>
<td>0.11</td>
</tr>
</tbody>
</table>
The steady state is approached. The ability in modeling the observed shear stress profile as the new characteristic slip distances, i.e., 

\[ \theta_p = \theta_1 + \theta_2 \]  

In order to describe the evolution of the state variables \( \theta_c \) and \( \theta_p \) with slip distance, \( \delta \), the following evolutionary laws for the state variables \( \theta_c \), \( \theta_1 \) and \( \theta_2 \) are used

\[ \frac{d\theta_c}{d\delta} = \frac{1}{L_v} \left( \theta_c + B \ln \left( \frac{V}{V'} \right) \right) \]  

\[ \frac{d\theta_1}{d\delta} = \frac{1}{L_1} \left( \theta_1 - CP \right) \]  

\[ \frac{d\theta_2}{d\delta} = \frac{1}{L_2} \left( \theta_2 - DP \right) \]  

Here it is worth mentioning that the necessity of introducing two characteristic slip distances, i.e., \( L_1 \) and \( L_2 \), was to provide flexibility in modeling the observed shear stress profile as the new steady state is approached.

Solutions to the evolutionary Eqs. (6)–(8) are given as

\[ \theta_c = B \ln \left( \frac{V}{V'} \right) + \left[ \theta_{c0} + B \ln \left( \frac{V}{V'} \right) \right] \exp \left( \frac{\delta_v - \delta}{L_v} \right) \]  

In each state, the steady-state frictional resistance of the interface corresponds to

\[ \frac{d\theta_i}{dt} = 0 \]  

for \( i = V, 1, 2 \) in Eqs. (6)–(8), respectively, to obtain

\[ \theta_v^s = -B \ln \left( \frac{V}{V'} \right) \]  

and

\[ \theta_i^s = \theta_i^s + \theta_i^s \]  

where

\[ \theta_i^s = CP \]  

and

\[ \theta_i^s = DP \]  

From these steady-state relations, the steady-state friction resistance of the interface can be expressed as

\[ \tau^s = \left[ \mu^s \theta_i^s + A \ln \left( \frac{V}{V'} \right) \right] \theta_p^s \]  

or

\[ \tau^s = \left[ \mu^s + (A - B) \ln \left( \frac{V}{V'} \right) \right] (C + D) P \]  

Constants \( A \) and \( B \) satisfy the inequality \( A < B \). This ensures the slip weakening effect, i.e., the interfacial shear resistance decreases with increasing slip velocity [43].

Constants \( C \) and \( D \) were chosen such that \( (C + D) = 1 \). This ensures that the steady-state coefficient of friction, \( \mu^s \), is independent of the normal pressure [43]. After a step drop in normal pressure, the state of the interface evolves toward another steady state. Therefore, constants \( C \) and \( D \) were selected such that \( (C + D) > 1 \). This made the steady-state value of kinetic friction larger in State 2 than in State 1, as was observed in the plate impact experiments on kinetic friction by Irfan and Prakash [28].
4 Modeling Using State and Rate Dependent Friction Model

The experimental results for Shot 9702 and Shot 9704 were used to validate the predictions of the state and rate dependent friction model described in Sec. 3. In the following, results of the two experiments and the predictions of the state and rate dependent friction model are presented. In order to obtain a good correlation between the experimental results and the predictions of the aforementioned state and rate dependent friction model, the following values of the constants $A-D$, the reference slip velocity $V^*$, the reference coefficient of friction $\mu^*$, and characteristic slip distances $L_1$, $L_2$, and $L_3$ were used: $A=0.011$, $B=0.0125$. Constants $A$ and $B$ are chosen such as to satisfy the inequality $A>B$, in order to ensure the slip weakening effect. $C=0.6$, $D=0.7$, for State 2. The constants $C$ and $D$ were chosen such that $(C+D) > 1$; $V^*=50$ m/s; $\mu^*=0.14$; $L_1=1 \mu$m, $L_2=6 \mu$m, and $L_3=1 \mu$m. The value of $L_3$ is larger than the value of $L_1$, since from the experimental results the approach of friction stress to the new steady state appears to be fast initially and slows later on.

Figures 2–4 show the correlation between the experimental results and the model predictions for Shot 9702. Figure 2 shows the measured interfacial friction stress as a function of time after impact. In this experiment both a step change in normal pressure as well as a step change in slip velocity were applied during the slip event. At impact, in State 1 the interfacial normal stress, $P_1$, was 1.3 GPa. Upon application of the pressure–shear impact loading the friction stress jumps to a steady-state level of $\tau_1=0.18$ GPa. This steady state continues for 1.8 $\mu$s, after which a step drop in normal stress from $P_1$ to $P_2$ is introduced. During the transition from State 1 to State 2, the normal stress $P_2$ is instantaneously reduced to 0.325 GPa; the shear stress $\tau_2$ does not follow the step drop in normal pressure. In fact, there is a gradual evolution of the shear-stress to a steady-state level. State 2 continues until approximately 3.44 $\mu$s, when a sudden change in the applied shear stress (at a constant normal pressure) is applied at the slip interface; this drop in applied shear stress results in a step drop in slip velocity from $V_1$ to $V_2$. Corresponding to this drop in slip velocity, the friction stress decreases rapidly initially, and is then followed by a slow rise in shear stress to a new steady-state level of $\tau_2=60$ MPa. Because of the relatively short window time for State 3 (only 0.3 $\mu$s), the final steady-state level for State 3 is difficult to interpret from the experiment results.

Figure 3 represents the interfacial shear stress as a function of the accumulated slip. For the time duration of the experiment ($\sim 3.74$ $\mu$s), the total accumulated slip is on the order of 150 $\mu$m. In State 1 the steady-state friction stress is $\tau_1=0.18$ GPa. The steady state continues until approximately 61.2 $\mu$m when a step drop in normal stress from $P_1$ to $P_2$ is introduced. This results in a gradual evolution of the friction stress to a new steady-state level. This new steady state, called State 2, continues until approximately 135.5 $\mu$m when a step drop in slip velocity from $V_2$ to $V_3$ occurs at a constant normal pressure $P_2$. This results in an initial rapid decrease in the friction stress followed by a slow rise in friction stress to a steady-state level of $\tau_2$.

Figure 4 shows the evolution of the coefficient of kinetic friction, $\mu_k$, with slip for Shot 9702. The level of the coefficient of friction in State 1 is approximately 0.14. During the transition from State 1 to State 2, there is an instantaneous jump in the coefficient of kinetic friction, followed by a new steady-state level. The jump in $\mu_k$ is because during the step drop in normal pressure (from State 1 to State 2), the friction stress does not follow the step drop in normal stress but rather evolves gradually to a lower steady-state value. This leads to an instantaneous jump in $\mu_k$, and as the slip continues $\mu_k$ decays exponentially to a steady-state level. The steady state level in State 2 is slightly higher from the steady-state level in State 1. This increase in $\mu_k$ can be attributed to perhaps a larger area of contact due to thermal softening of the asperities at the slip surfaces immediately following the step drop in normal stress. However, the transition from State 1 to State 2 clearly reveals a much stronger dependence of the frictional stress to changes of normal pressure. First, there is an instantaneous jump in the value of $\mu_k$ as the normal pressure is changed instantaneously from $P_1$ to $P_2$, and second, the steady state $\mu_k$ shows a dependence on the normal stress in the two states.

Figures 5–7 show the correlation between the experimental results and the model predictions for Shot 9704. Figure 5 shows the measured friction stress as a function of time after impact. During State 1, the normal stress, $P_1$, at the interface was 1.078 GPa. Immediately after impact the friction stress jumps to a steady-state level characteristic of the interfacial condition in State 1. In this steady state the friction stress $\tau_1$ is $\sim 0.365$ GPa. The steady state continues to 1.8 $\mu$s, when a step drop in normal stress (from $P_1$ to $P_2$) is introduced. During the transition from State 1 to State 2, the normal pressure $P_2$ reduces to 0.27 GPa instantaneously, whereas the friction stress $\tau_2$ does not follow the step drop in normal pressure. In fact, there is a gradual evolution of the shear stress to a new steady-state level. State 2 continues until approximately 3.44 $\mu$s when a change in the applied shear stress at the interface (at a constant normal pressure) results in a step drop in slip velocity from $V_2$ to $V_3$. This results in an initial rapid decrease in the friction stress followed by a slow rise in shear stress to a steady-state level $\tau_2$. As stated earlier, this final steady-state level in State 3 is difficult to interpret from the present experimental results because of the relatively short time window for State 3 (only 0.3 $\mu$s).

Figure 6 shows the friction stress as a function of accumulated slip. The total accumulated slip in the experiment was 70 $\mu$m. In State 1, the frictional interface is in a steady state; the friction stress level is $\tau_1$. This steady state continues until approximately 13 $\mu$m when a step drop in normal pressure from $P_1$ to $P_2$ is introduced. During the transition from State 1 to State 2, the normal stress $P_2$ is instantaneously reduced to 0.325 GPa; the shear stress $\tau_2$ does not follow the step drop in normal pressure. This results in an initial rapid decrease in the friction stress followed by a slow rise in friction stress to a steady-state level of $\tau_2$.

Figure 7 shows the evolution of the coefficient of kinetic friction $\mu_k$ for shot 9704 with accumulated slip. In State 1, the value of coefficient of kinetic friction is 0.34. In State 2, the value of $\mu_k$ rises after the step drop in normal pressure and then decays to a steady-state value. In State 3, as a result of step drop in slip velocity, the coefficient of kinetic friction $\mu_k$ decreases rapidly and then evolves toward a steady-state level.

The state and the rate dependent friction model with the aforementioned values for the parameters appear to capture the transients in the experimental results for Shots 9702 and 9704 adequately. The experiments show that during the transition from State 1 to State 2, the normal stress is instantaneously reduced while the shear stress $\tau_1$ does not follow the step drop in normal pressure; rather there is a gradual evolution of the shear stress to a steady-state level. Also the step drop in slip velocity from $V_2$ to $V_3$ results in an initial rapid decrease in the friction stress followed by a slow rise in friction stress to a steady-state level of $\tau_3$. The mathematical model clearly depicts these transitions and evolution to new steady states.

5 Summary

A good correlation was found between the experimental graphs and the mathematical models, i.e.: (a) a step drop in normal pressure gave an exponential decay of the shear stress to a new steady state, characteristic of the current normal pressure and the current slip velocity; and (b) a step drop in interfacial slip velocity led to an initial drop in transmitted shear stress, which later increased toward a new steady-state shear stress value. It was also observed that large changes in slip velocity led to relatively small changes.
in the steady-state shear stress level. The steady state shear stress \( \tau \) was found logarithmically dependent on the slip velocity \( V \). The transient shear stress has a positive dependence on instantaneous slip velocity, i.e., a step increase in \( \tau \) was observed when the slip velocity was suddenly increased. Similarly a step decrease in \( \tau \) was observed when the slip velocity was suddenly decreased. The characteristic displacements associated with the normal pressure and accumulated slip displacement were about the same magnitudes as that observed in the velocity stepping. Two state variables, \( \theta_t \) and \( \theta_v \), were required to correctly model the behavior of the interface subjected to changes in velocity and pressure, respectively.

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**Nomenclature**

- \( \mu \) = a constant reference value of the coefficient of friction
- \( V' \) = a reference value of the slip velocity
- \( \delta_{sp} \) = slip displacement prior to step decrease or increase in normal pressure
- \( \delta_{ss} \) = slip displacement prior to step increase or decrease in slip velocity
- \( L_c \) = characteristic distance for the exponential decay in \( \theta_t \)
- \( L_1 \) = characteristic distance for the exponential decay in \( \theta_t \) as slip occurs
- \( L_2 \) = characteristic distance for the exponential decay in \( \theta_t \) as slip occurs
- \( A \) = constant for step changes in slip velocity
- \( B \) = constant for step changes in slip velocity
- \( C \) = constant for step changes in normal pressure
- \( D \) = constant for step changes in normal pressure
- \( \theta_{sp} \) = a state variable introduced to describe the changes in state resulting from changes in the normal pressure
- \( \theta_{ss} \) = a state variable introduced to describe the changes in state resulting from changes in the slip velocity
- \( \theta_2 \) = a state variable which characterizes the small-slip transition regime following a change in the normal pressure
- \( \theta_1 \) = a state variable which characterizes the large-slip transition regime following a change in the normal pressure
- \( \theta_{sp} \) = value of state variable prior to step decrease or step increase in normal pressure
- \( \theta_{ss} \) = value of state variable prior to step decrease or step increase in slip velocity

**References**